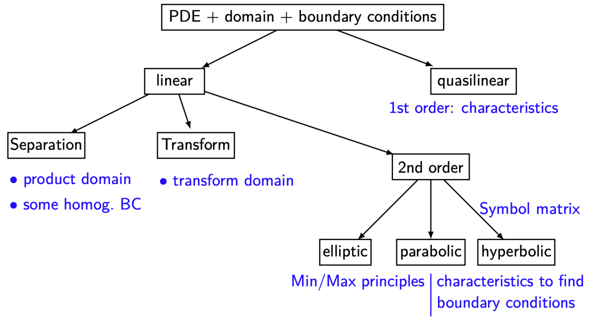
Partial Differential Equations

Domain and Boundary

- Domain Ω is an open subset of $\mathbb{R}^n$ (meaning all points are interior points)
- Boundary has to meet conditions:
- *Dirichlet boundary conditions*: specify values u on $\partial\Omega$: $u(x) = f(x)\ \forall x \in \partial\Omega$
- *Neumann boundary conditions*: specify derivatives of u on boundary. Only derivatives orthogonal to the boundary give additional information: normal derivative: $\frac{\partial u}{\partial n} = g(x)\ \forall x \in \partial\Omega$

Classification



Linearity

Given an equation involving a function $u(x), x \in \mathbb{R}$ and its derivatives, there is a function F describing the relation:

$$F\left(\begin{array}{c} x, y, z, p, q, s, t, r, \ldots \\ \downarrow \text{(corresponding to)} \downarrow \\ x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}, \ldots \end{array}\right) = 0$$

(common variable names $p_i \to \frac{\partial u}{\partial x_i}$ and $t_{ij} \to \frac{\partial^2 u}{\partial x_i \partial x_j}$)
A PDE is *linear* when function F is linear in $u, p_1, \ldots, p_n, t_{11}, \ldots, t_{nn}, \ldots$.
A PDE is *quasilinear* when function F is linear in $p_1, \ldots, p_n, t_{11}, \ldots, t_{nn}, \ldots$.
For example, given the heat equation $u_t = \kappa u_{xx}$, F would be $F(p_1, t_{22}) = p_1 - \kappa t_{22}$.

2$^{\text{nd}}$ Order PDEs: Symbol Matrix

The symbol matrix of the 2$^{\text{nd}}$ order partial differential operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

is the symmetric matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

For example:

$$L = a\frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2} + 3\frac{\partial^2}{\partial z^2} + \alpha\frac{\partial^2}{\partial x\,\partial y} - 1291\frac{\partial^2}{\partial x\,\partial z} + 47\frac{\partial^2}{\partial z\,\partial y} - c19\frac{\partial}{\partial x} + 5$$

$$A = \begin{array}{cc} \left(\begin{array}{ccc} a & \frac{\alpha}{2} & -\frac{1291}{2} \\ \frac{\alpha}{2} & x & \frac{47}{2} \\ -\frac{1291}{2} & \frac{47}{2} & 3 \end{array}\right) & \begin{array}{c} x \\ y \\ z \end{array} \\ \begin{array}{ccc} x & y & z \end{array} & \end{array}$$

The type of equation can be inferred by the sign of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the symbol matrix. Calculating the determinant ($\det A = \prod_i \lambda_i$) and trace ($\operatorname{tr} A = \sum_i \lambda_i$) reveals information about the signs of its eigenvalues:

Two Variables of PDE

$$\det A \begin{cases} > 0 & \text{elliptic (eigenvalues have same sign)} \\ = 0 & \text{parabolic (at least one eigenvalue is zero)} \\ < 0 & \text{hyperbolic (eigenvalues have opposite sign)} \end{cases}$$

Three Variables of PDE

$$\begin{aligned} \operatorname{rank} A < 2 &\Rightarrow \text{not classified} \\ \det A \text{ and } \operatorname{tr} A \text{ have different sign} &\Rightarrow \text{hyperbolic} \\ \det A = 0, \text{ semidefinite (Cholesky decomp.)} &\Rightarrow \text{parabolic} \\ A \text{ or } -A \text{ positive definite (Cholesky)} &\Rightarrow \text{elliptic} \\ \text{all other cases} &\Rightarrow \text{hyperbolic} \end{aligned}$$

Quasilinear PDEs

Characteristics

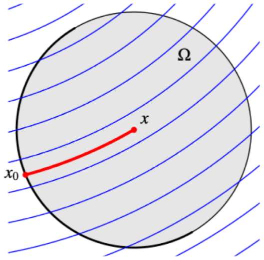
The PDE

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

can be written in vector notation:

$$\underbrace{\begin{pmatrix} a(x,y,u) \\ b(x,y,u) \\ c(x,y,u) \end{pmatrix}}_{\vec{t}} \cdot \underbrace{\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ -1 \end{pmatrix}}_{\vec{n}} = 0$$

where $\vec{n}$ is a normal vector and $\vec{t}$ is always tangential to the solution surface. Therefore, we can elaborate a solution algorithm:



1. Using the Cauchy initial curve, we formulate

   $$\vec{v}(s) = \begin{pmatrix} v_x(s) & v_y(s) & v_z(s) \end{pmatrix}^T$$

   which is a point on the initial curve, parameterised by s.
2. Find characteristic curves as solution of the ODEs

   $$\frac{d}{dt}\begin{bmatrix} x(t,s) \\ y(t,s) \\ z(t,s) \end{bmatrix} = \begin{bmatrix} a(x(t,s), y(t,s), z(t,s)) \\ b(x(t,s), y(t,s), z(t,s)) \\ c(x(t,s), y(t,s), z(t,s)) \end{bmatrix}$$

   with

   $$\begin{bmatrix} x(0,s) \\ y(0,s) \\ z(0,s) \end{bmatrix} = \vec{v}(s) = \begin{bmatrix} v_x(s) \\ v_y(s) \\ v_z(s) \end{bmatrix}$$

3. Eliminate the variables t and s and condense solution into a function $u(x,y)$:

   $$\left.\begin{aligned} x &= x(t,s) \\ y &= y(t,s) \\ u &= z(t,s) \end{aligned}\right\} \quad u = u(x,y)$$

Linear PDEs

Separation

Conditions, when separation may be successful:

- Homogeneous, linear PDE
- Homogeneous boundary conditions
- Domain must be a cartesian product (i.e. some form of rectangle when in cartesian coordinate system)

if these are given, we can try:

1. Assume the structure of the solution to be in the form of some ansatz with separable variables, usually a product $u(x,y) = X(x) \cdot Y(y)$
2. Substitute u in PDE with ansatz by variable:

   all terms of x = all terms of $y = \lambda$

   and solve the ordinary differential equations for $X(x)$ and $Y(y)$.
3. Use *homogeneous* boundary conditions to determine admissible values λ_k.
4. Solve equations: $X_k(x)$ and $Y_k(y)$
   $\Rightarrow u_k(x,y) = X_k(x) \cdot Y_k(y)$ using ansatz.
5. Combine solutions: $u(x,y) = \sum_{k\in\mathbb{Z}} (a_k u_k(x,y))$
6. Use remaining boundary conditions to determine a_k.

Separation Example Given the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ for a string mounted between $u(0,t) = u(\pi,t) = 0$ and is in the rest position at $t = 0$: $u(x,0) = 0$ but has initial velocity $\frac{\partial u}{\partial t}(0,x) = \sin^3(x) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$. We choose the Ansatz $u(x,t) = X(x)T(t)$:

$$\begin{aligned} X(x)T''(t) &= X''(x)T(x) \qquad\qquad | \div X(x)T(t) \\ T''(t) \div T(t) &= X''(x) \div X(x) = \mu \end{aligned}$$

Hence, we receive the ODEs

$$\begin{aligned} \mu T(t) &= T''(t) \\ \mu X(x) &= X''(x) \end{aligned}$$

with $\mu < 0$ to receive oscillating solutions, the ODEs have the solutions

$$\begin{aligned} X(x) &= C\sin(\sqrt{\mu}x) + \cancel{D\cos(\sqrt{\mu}x)} \\ T(t) &= A\sin(\sqrt{\mu}t) + B\cos(\sqrt{\mu}t) \end{aligned}$$

by taking into account the boundary conditions $X(0) = X(\pi) = 0$, yielding $C = 1, D = 0$ and $\sqrt{n} \in \mathbb{N}$. Therefore, we have the general solution

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \sin(nx) \cdot (A_n \sin(nt) + B_n \cos(nt)) \\ &= \sum_{n=1}^{\infty} A_n \sin(nx)\sin(nt) + \sum_{n=1}^{\infty} B_n \sin(nx)\cos(nt) \end{aligned}$$

considering the boundary conditions:

$$u(x,0) = 0 = \sum_{n=1}^{\infty} B_n \sin(nx) \underbrace{\cos(0)}_{=1} \Rightarrow B_n = 0$$

and

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} nA_n \sin(nx)\cos(nt)$$

$$\rightarrow \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} nA_n \sin(nx) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

$$\Rightarrow A_1 = 3/4, A_3 = -1/12, A_{n\setminus\{1,3\}} = 0$$

$$\Rightarrow u(x,t) = (3/4)\sin(x)\sin(t) - (1/12)\sin(3x)\sin(3t)$$

Transformation

1. Transform equation in the unbounded variable (e.g. t): derivatives turn into algebraic expressions:
$$\left(\mathcal{L}\frac{\partial u(t,y)}{\partial t}\right)(s) = s\underbrace{(\mathcal{L}f)(s,y)}_{=y_s(y)} - u(0,y)$$
2. Transform y-boundary conditions: gives boundary conditions for
$$\begin{aligned} u(t,0) &= g(t) \\ y_s(0) &= (\mathcal{L}u(t,0))(s) = (\mathcal{L}g)(s) \end{aligned}$$
3. Solve PDE with fewer derivatives, ODEs
4. Inverse transform

In general, one could use the following transforms:

Domain	Transform
$[-\pi,\pi]$	Fourier Series
$[a,b]$	Fourier Series
$\mathbb{R}$	Fourier Transform
$\mathbb{R}_+$	Laplace Transform
G	Generalised Fourier Theory

Laplace Transform

(Only works on linear equations.) The Laplace transform of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is the function

$$\mathcal{L}f : \mathbb{R}^+ \to \mathbb{R} : s \rightarrowtail \mathcal{L}f(s) = \int_0^{\infty} f(t)e^{-st}\, dt$$

It is linear:

$$\mathcal{L}(\alpha f + \beta f) = \alpha\mathcal{L}f + \beta\mathcal{L}f$$

Example transformations:

Constant:	Exponential:	Derivative:
$f(t) = c$	$f(t) = e^{-ct}$	$f(t) = g^{(n)}(t)$
$(\mathcal{L}f)(s) = \frac{c}{s}$	$(\mathcal{L}f)(s) = \frac{1}{c+s}$	$(\mathcal{L}f^{(n)})(s) =$
		$-f^{(n-1)}(0) + s\left(\mathcal{L}f^{(n-1)}\right)(s)$
		(removes t-derivatives: $\frac{\partial}{\partial t} \to s$)

Fourier Transform

For $f : \mathbb{R} \to \mathbb{C} : x \rightarrowtail f(x)$ the Fourier transform of f is defined as

$$\mathcal{F}f = \hat{f} : \mathbb{R} \to \mathbb{C} : k \rightarrowtail \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx$$

It turns the derivative $\frac{\partial}{\partial x}$ into a multiplication by $-ik$ (second derivatives are reduced to $i^2 = -1$):

$$(\mathcal{F}f^{(n)})(k) = (ik)^n \mathcal{F}f(k)$$

The function f can be recovered from $\hat{f}$ by

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\, dk$$

PDEs of Second Order

Linear PDEs of second order have the form

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} u + cu = f$$

The equations fall into these categories: *Elliptic* (potential problem), *parabolic* (heat equation, diffusion) and *hyperbolic* (wave equation, linearised supersonic flow).

Splitting the Solution

Given a second order linear differential operator L, we have the PDE $Lu = f$ in Ω with $u = g$ on $\partial\Omega$.

1. We try to find a particular solution $Lu_p = f$ in Ω, satisfying only the PDE and neglecting boundary conditions
2. To solve the original problem, we need an additional summand u_r taking care of boundary values to receive the solution $u_p + u_r$. However, u_r still needs to solve the PDE but due to linearity, this reduces to a homogeneous problem:
$$L(u_p + u_r) = f + Lu_r = f \Rightarrow Lu_r = 0 \text{ in } \Omega$$
3. Ensure that the boundary conditions are satisfied:
$$u_p + u_r = g \Rightarrow u_r = g - u_p \text{ on } \partial\Omega$$
4. If the solution is not unique, we're able to find other solutions using an additional term u_h:
$$\begin{aligned} L(u_p + u_r + u_h) &= f + Lu_h &\Rightarrow Lu_h &= 0 \text{ in } \Omega \\ u_p + u_r + u_h &= g + u_h &\Rightarrow u_h &= 0 \text{ on } \partial\Omega \end{aligned}$$

For example, consider the PDE $\nabla^2 u = 4$ in $\Omega = \{(x,y) \mid x^2 + y^2 < 1\}$ and $u = 0.5x + 0.5$ on $\partial\Omega$. This has the particular solution $u_p(x,y) = x^2 + y^2$. To fix boundary conditions, we find a solution u_r of the homogeneous problem with boundary values $u_r = 0.5x + 0.5 - u_p(x,y) = 0.5x + 0.5 - \underbrace{1}_{\text{on } \partial\Omega} = 0.5x - 0.5$ giving the complete solution $u(x,y) = u_p(x,y) + u_r(x,y) = x^2 + y^2 + 0.5x - 0.5$.

Elliptic PDEs

Maximum Principle for Elliptic Operators

Theorem: If L is an elliptic differential operator on a connected and bounded domain Ω, and u is a solution $Lu = 0$, then u takes its maximum and minimum on the boundary of Ω.

Uniqueness of Solutions

Theorem: If L is an elliptic differential operator on a connected and bounded domain Ω, then there is at most one solution of $Lu = f$ with boundary conditions $u_{|\partial\Omega} = g$.

Green's Function

Heaviside function: $\vartheta(x - \xi) = \begin{cases} 1 & \xi \leq x \\ 0 & \xi > x \end{cases}$ translates the integral $\int_0^x f(\xi)d\xi$ to $\int_0^1 \vartheta(x - \xi) \cdot f(\xi)d\xi$ for ξ in $[0,1]$.

Given the Laplace equation $\nabla^2 u = f$, we want to find an "inverse" Laplace such that

$$u(x) = \int G(x,\xi)f(\xi)\, d\xi$$

There is a function $G(x,\xi)$ on $\bar{\Omega} \times \bar{\Omega}$ called Green's function such that

$$\Delta G(x,\xi) = \delta(x - \xi) \quad \text{in } \Omega \text{ and } G(x,\xi) = 0 \text{ for } \xi \in \partial\Omega$$

the general Poisson problem $\Delta u = f$ in Ω with boundary conditions $u = g$ on $\partial\Omega$ has the solution

$$u(x) = \int_\Omega G(x,\xi) f(\xi)\ d\xi + \int_{\partial\Omega} g(\xi) \cdot \mathrm{grad}_\xi G(x,\xi)\ dn$$

with n is an outside pointing normal. To construct a particular solution u_p of $\Delta u = f$ on $\Omega \subset \mathbb{R}^n$, there are the following Green's functions (using Dirac-δ function):

$$G(x,\xi) = \left\{ \begin{array}{ll} \frac{1}{2}|x-\xi| & \text{for } n=1 \\ \frac{1}{2\pi}\log|x-\xi| & \text{for } n=2 \\ \frac{1}{4\pi}\frac{1}{|x-\xi|} & \text{for } n=3 \end{array} \right\} \Rightarrow \Delta G(x,\xi) = \delta(x-\xi)$$

then $u_p(x) = \int_\Omega G(x,\xi) f(\xi)\ d\xi$.

Hyperbolic PDEs

D'Alembert Solution of Wave Equation

The wave equation can be factorized:

$$0 = \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right) u$$

giving two quasilinear PDEs of which the superposition is a solution:
$u(t,x) = u_+(x-at) + u_-(x+at)$

Strip and Characteristics

Consider the hyperbolic equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = g - d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = h.$$

A strip then is a curve $(x(s), y(s), u(x))$ together with the slopes of the tangent planes $p(s) = \frac{\partial u}{\partial x}$ and $q(s) = \frac{\partial u}{\partial y}$.
Consequence: A solution $u(t,x)$ of the wave equation determines a strip for each value t:
$x(s) = s \quad t(s) = t \quad u(s) = u(t,s) \quad p(s) = \frac{\partial u}{\partial x}(s,t) \quad q(s) = \frac{\partial u}{\partial t}(s,t)$
with an initial strip
$x(s) = s \quad t(s) = 0 \quad u(s) = f(s) \quad p(s) = f'(s) \quad q(s) = g(s)$
the Cauchy-Problem therefore has to be formulated in terms of the strip.
The second partial derivatives are then determined by the linear system

$$\begin{aligned} a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} &= h(s) \ = g - dp(s) - eq(s) - fu \\ \dot{x}(s)\frac{\partial^2 u}{\partial x^2} + \dot{y}(s)\frac{\partial^2 u}{\partial x \partial y} &= \dot{p}(s) \\ \dot{x}(s)\frac{\partial^2 u}{\partial x \partial y} + \dot{y}(s)\frac{\partial^2 u}{\partial y^2} &= \dot{q}(s) \end{aligned}$$

The *characteristics* of a differential equation are the curves $t \mapsto (x(t), y(t))$ for which the initial data does not determine the second partial derivatives uniquely and thus, determinant of the linear system is zero

$$\det \begin{bmatrix} a & 2b & c \\ \dot{x}(s) & \dot{y}(s) & 0 \\ 0 & \dot{x}(s) & \dot{y}(s) \end{bmatrix} = 0$$

The determinant is constructible from the *symbol matrix A* and we thus receive a differential equation for the characteristics:

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \Rightarrow \quad a\dot{y}(s)^2 - 2b\dot{x}(s)\dot{y}(s) + c\dot{x}(s)^2 = 0$$

(note: t is the mapping variable of the curve, if the PDE uses time (e.g. wave) s may be used)
The equation for the characteristics can be simplified for non-constant by dividing by e.g. $\dot{x}(s)^2$. For example:

$$\begin{aligned} &x\dot{y}(s)^2 + \dot{x}(s)\dot{y}(s) + \dot{x}(s)^2 = 0 \quad | \div \dot{x}(s)^2 \\ &= x\frac{\dot{y}(s)^2}{\dot{x}(s)^2} + \frac{\dot{y}(s)}{\dot{x}(s)} + 1 = 0 \quad \Big| \ \dot{y}/\dot{x} = \frac{dy/ds}{dx/ds} = y' \\ &= x(y')^2 + y' + 1 = 0 \end{aligned}$$

which is a quadratic first order differential equation that can be factored into two first order linear differential equations.
Property: Touch Characteristics touch if the equation

$$a(y')^2 - 2by' + c = 0$$

has *one solution* for y'. The different solutions intersect, if there are multiple solutions for y' ($0 \neq (-2b)^2 - 4ac = -4\det A > 0$).
$\Rightarrow$ Different solution surfaces touch along characteristic curves.
Property: Influence We can find the boundaries influencing a certain point P by first finding the characteristic curves that intersect the point by solving the ODEs and setting the constants based on x- and y-values of P and afterwards, calculating using the curves where they start on the boundary.
Summary Characteristic curves are the curves that are not possible as Cauchy initial curve and describe the evolution of a PDE.

Numerical Methods

Discretisation of Operators

$$\frac{\partial g}{\partial x} \approx \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$\frac{\partial^2 g}{\partial x^2} \approx \frac{g(x+\Delta x) - 2\cdot g(x) + g(x-\Delta x)}{\Delta x^2}$$

(Δ referring to step size)

Discrete Laplace-Operator / Five-Point-Star Operator

Setting $h = \Delta x = \Delta y$, then $\nabla^2 u$ is

$$\frac{1}{h^2}\left(\underbrace{u(x+h,y)}_{\text{East}} + \underbrace{u(x,y+h)}_{\text{North}} + \underbrace{u(x-h,y)}_{\text{West}} + \underbrace{u(x,y-h)}_{\text{South}} - \underbrace{4u(x,y)}_{\text{Center}}\right)$$

Finite Difference Method for Elliptic PDEs

In case of a one-dimensional domain:

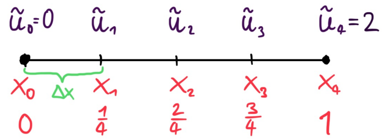
1. Given the differential equation $Lu(x) = f(x)$
2. Discretise domain. E.g. we have $x_k^{(n)} = k \cdot \Delta x$ with $\Delta x = 1/n$.
3. Discretise equation using discrete operators
4. Replace functions $u(x)$ and $f(x)$ by vectors of nodal values: $u_k^{(n)} = u\left(x_k^{(n)}\right)$ and $f_k^{(n)} = f\left(x_k^{(n)}\right)$
5. Solve for $A^{(n)} \cdot \tilde{u}^{(n)} = f^{(n)}$ for inner knots while considering boundary conditions.

One-Dimensional Example

The boundary value problem
$u''(x) = 4 \cdot (u(x) - x) = 4u(x) - 4x, \quad x \in]0,1[$ with $u(0) = 0$ and $u(1) = 2$ should be approximated by function $\tilde{u}(x)$ using FDM with $\Delta x = 1/4$.

The discretised equation therefore is:

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\Delta x^2} = 4u_k - 4x_k \quad (k = 1, \ldots, n-1)$$



giving us the linear system

$$\left| \begin{aligned} \frac{\tilde{u}_2 - 2\tilde{u}_1 + \cancelto{=0}{\tilde{u}_0}}{\cancelto{=1/16}{\Delta x^2}} &= 4\tilde{u}_1 - 4\cancelto{1/4}{x_1} \\ \frac{\tilde{u}_3 - 2\tilde{u}_2 + \tilde{u}_1}{\cancelto{=1/16}{\Delta x^2}} &= 4\tilde{u}_2 - 4\cancelto{2/4}{x_2} \\ \frac{\cancelto{=2}{\tilde{u}_4} - 2\tilde{u}_3 + \tilde{u}_2}{\cancelto{=1/16}{\Delta x^2}} &= 4\tilde{u}_3 - 4\cancelto{3/4}{x_3} \end{aligned} \right|$$

resulting in

$$\begin{aligned} -32\tilde{u}_1 + 16\tilde{u}_2 &= 4\tilde{u}_1 - 1 & &\rightarrow -36\tilde{u}_1 + 16\tilde{u}_2 = -1 \\ 16\tilde{u}_1 - 32\tilde{u}_2 + 16\tilde{u}_3 &= 4\tilde{u}_2 - 2 & &\rightarrow 16\tilde{u}_1 - 36\tilde{u}_2 + 16\tilde{u}_3 = -2 \\ 16\tilde{u}_2 - 32\tilde{u}_3 + 16 \cdot 2 &= 4\tilde{u}_3 - 3 & &\rightarrow 16\tilde{u}_2 - 36\tilde{u}_3 = -35 \end{aligned}$$

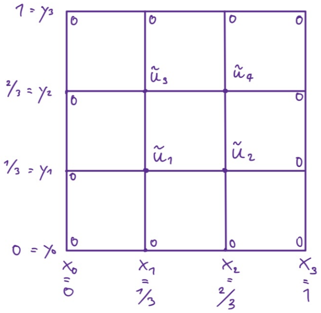
ultimately giving the equation in matrix form (optional):

$$\begin{bmatrix} -36 & 16 & 0 \\ 16 & -36 & 16 \\ 0 & 16 & -36 \end{bmatrix} \cdot \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -35 \end{bmatrix}$$

giving the approximation vector $\tilde{u} \approx (0.4, 0.83, 1.34)$.

Two-Dimensions Example

We have Poisson's differential equation $-\Delta u(x,y) = f(x,y)$ with $f(x,y) = \left((3x+x^2)\cdot y(1-y) + (3y+y^2)\cdot x(1-x)\right)\cdot e^{x+y}$ on $\Omega = [0,1]\times[0,1]$ with homogeneous boundary conditions. Setting $h = \Delta x = \Delta y = \frac{1}{3}$, we can use the discrete Laplace operator. Discretisation of the geometry yields



and we can define for each inner node an equation:

$$u_1 : -\frac{\tilde u_2 + \tilde u_3 + \cancelto{0}{y_1} + \cancelto{0}{x_1} - 4\tilde u_1}{\cancelto{(1/3)^2}{1/h^2}} = f(1/3, 1/3) = f_1$$

$$u_2 : -\frac{0 + \tilde u_4 + \tilde u_1 + 0 - 4\tilde u_2}{(1/3)^2} = f(2/3, 1/3) = f_2$$

$$u_3 : -\frac{\tilde u_4 + 0 + 0 + \tilde u_1 - 4\tilde u_3}{(1/3)^2} = f(1/3, 2/3) = f_3$$

$$u_4 : -\frac{0 + 0 + \tilde u_3 + \tilde u_2 - 4\tilde u_4}{(1/3)^2} = f(2/3, 2/3) = f_4$$

giving the linear system:

$$-9\cdot\begin{bmatrix}-4 & 1 & 1 & 0\\ 1 & -4 & 0 & 1\\ 1 & 0 & -4 & 1\\ 0 & 1 & 1 & -4\end{bmatrix}\begin{bmatrix}\tilde u_1\\ \tilde u_2\\ \tilde u_3\\ \tilde u_4\end{bmatrix} = \begin{bmatrix}f_1\\ f_2\\ f_3\\ f_4\end{bmatrix}$$

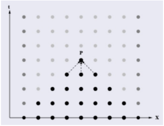
Finite Differences Method for Parabolic PDEs

Richardson's *Explicit FD Scheme*

1. *Discretise parabolic PDE* $Lu(\dots) = f(\dots)$ using discrete operators
2. *Discretise geometry* by introducing a grid: $x_{j,k} = (j\cdot\Delta x, k\cdot\Delta t)$ with j as the local index and k as the time index where $k=0$ is on boundary and $\Delta x = \frac{1}{n}$, $\Delta t = \frac{r}{n^2}$ and $r = \frac{\Delta t}{\Delta x^2}$
3. *Introduce approximated nodal values* $\tilde u(x_{j,k}) = \tilde u_{j,k}$ and $f_j = f(x_j, 0)$ for the Dirichlet boundary conditions.
4. Find a matrix C that satisfies $\tilde u_j^{(k+1)} = r\tilde u_{j-1}^{(k)} + (1-2r)\tilde u_j^{(k)} + r\cdot\tilde u_{j+1}^{(k)}$ for inner grid points for each "step" satisfying boundary conditions: $\tilde u_j^{(0)} = \tilde u_{j,0} = f_j$
5. Iteratively generate solution vector using $\tilde u^{(k+1)} = C\cdot\tilde u^{(k)}$ (i.e. $\tilde u^{(k)} = C^k\cdot\vec f$)

Stability Analysis

The scheme of Richardson tends asymptotically to zero with k to infinity if for eigenvalues of C, $|\lambda_{\max}| < 1$ is true or $||C^{(n)}|| < 1$ with respect to the spectral norm of C. We only receive a good approximation if the steps are small enough to "catch" enough information on the boundary.



Example Using Heat Equation

Given the heat equation $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$ for the domain $\Omega = [0,1]\times[0,\infty[$ with the boundary conditions $u(x,0) = e^x$, $u(0,t) = e^t$ and $u(1,t) = e^{1+t}$, we want to find an approximation $\tilde u$ for points $(1/3, 2/3), (2/3, 2/3)$ using $\Delta x = 1/3$ and $\Delta t = 1/3$.

Discretisation of Heat Equation:

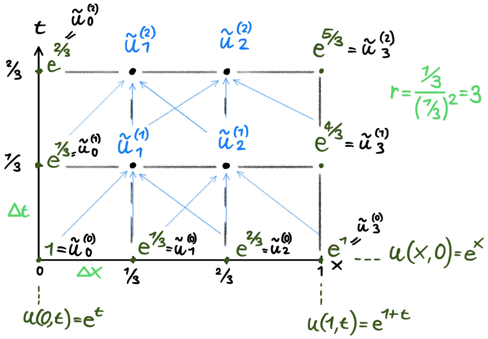
$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \approx \frac{u(x+\Delta x,t) - 2\cdot u(x,t) + u(x-\Delta x,t)}{\Delta x^2}$$

$$u(x,t+\Delta t) \approx u(x,t) + \Delta t\cdot\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2}$$

$$\tilde u_{j,k+1} = u_{j,k} + r\cdot u_{j+1,k} - 2u_{j,k} + u_{j-1,k} \quad \Big| \quad r = \frac{\Delta t}{\Delta x^2}$$

$$\tilde u_{j,k+1} = r\cdot\tilde u_{j-1,k} + (1-2r)\tilde u_{j,k} + r\cdot\tilde u_{j+1,k}$$

Discretisation of Geometry:



Solution Method: With r having a set value, we can write the iterative equation as:

$$\tilde u_j^{(k+1)} = 3\tilde u_{j-1}^{(k)} + 5\tilde u_j^{(k)} + 3\tilde u_{j+1}^{(k)}$$

and thus, for $\mathbf{k+1=1}$:

$$\tilde u_1^{(1)} = 3\cancelto{1}{\tilde u_0^{(0)}} - 5\cancelto{e^{1/3}}{\tilde u_1^{(0)}} + 3\cancelto{e^{2/3}}{\tilde u_2^{(0)}} = 1.8651$$

$$\tilde u_2^{(1)} = 3\tilde u_1^{(0)} - 5\tilde u_2^{(0)} + 3\tilde u_3^{(0)} = 2.603$$

and $\mathbf{k+1=2}$:

$$\tilde u_1^{(2)} = 3\cancelto{e^{\Delta t\cdot k}}{\tilde u_0^{(1)}} - 5\tilde u_1^{(1)} + 3\tilde u_2^{(1)} = 2.6702$$

$$\tilde u_2^{(2)} = 3\tilde u_1^{(1)} - 5\tilde u_2^{(1)} + 3\cancelto{e^{1+\Delta t\cdot k}}{\tilde u_3^{(1)}} = 3.9614$$

Alternatively, using matrix C:

$$\begin{bmatrix}\tilde u_1\\ \tilde u_2\end{bmatrix}^{(k+1)} = \underbrace{\begin{bmatrix}-5 & 3\\ 3 & -5\end{bmatrix}}_{C}\begin{bmatrix}\tilde u_1\\ \tilde u_2\end{bmatrix}^{(k)} + 3\underbrace{\begin{bmatrix}e^{\Delta tk} = e^{\frac{k}{3}}\\ e^{1+\Delta tk} = e^{1+\frac{k}{3}}\end{bmatrix}}_{\text{Dirichlet B.C.}}$$

and

$$\begin{bmatrix}\tilde u_1\\ \tilde u_2\end{bmatrix}^{(2)} = C^2\vec{\tilde u}^{(0)} + C\cdot 3\begin{bmatrix}1\\ e\end{bmatrix} + 3\begin{bmatrix}e^{1/3}\\ e^{4/3}\end{bmatrix} = \begin{bmatrix}2.6702\\ 3.9614\end{bmatrix}$$

Richardson's *Implicit Scheme*

Uses a backward finite difference for the discretisation in t:

$$\frac{\partial u}{\partial t} \approx \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t}$$

to be able to consider all boundary points. We therefore look for a matrix E to receive a system of linear equations:

$E\cdot\tilde u^{(k+1)} = \tilde u^{(k)}$

and get a solution by inverting (of course not suitable for numerical tasks, only theoretical):

$$\tilde u^{(k+1)} = \left(E^{(n)}\right)^{-1}\cdot\tilde u^{(k)}$$

The method is stable independently of r (absolute stability).

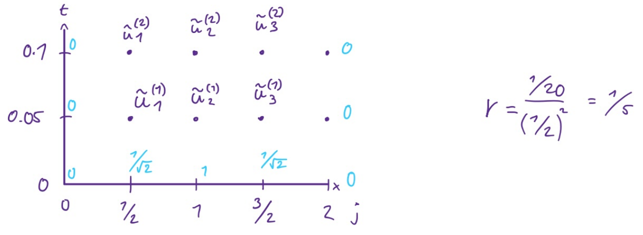
Example

Given the heat equation $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$ for the domain $\Omega = [0,2]\times[0,\infty[$ with boundary condition $u(x,0) = \sin\left(\frac{\pi x}{2}\right)$. We want to find $\tilde u(x, 0.1)$ using $\Delta x = 0.5$ and $\Delta t = 0.05$.

Discretisation of Heat Equation analogously like in the explicit method:

$$\tilde u_j^{(k)} = -r\cdot\tilde u_{j-1}^{(k+1)} + (1+2\cdot r)\cdot\tilde u_j^{(k+1)} - r\cdot\tilde u_{j+1}^{(k+1)}$$

Discretisation of Geometry



$f(x) = \sin(\pi/2 x)$

Solution

For $k=0$:

$$\mathbf{j=1} : -\cancelto{\frac{1}{5}}{r}\cdot\cancelto{0}{\tilde u_0^{(1)}} + 7/5\cdot\tilde u_1^{(1)} - \cancelto{\frac{1}{5}}{r}\cdot\tilde u_2^{(1)} = f(1/2) = 1/\sqrt{2}$$

$$\mathbf{j=2} : -\cancelto{\frac{1}{5}}{r}\cdot\tilde u_1^{(1)} + 7/5\cdot\tilde u_2^{(1)} - \cancelto{\frac{1}{5}}{r}\cdot\tilde u_3^{(1)} = f(1) = 1$$

$$\mathbf{j=3} : -\cancelto{\frac{1}{5}}{r}\cdot\tilde u_2^{(1)} + 7/5\cdot\tilde u_3^{(1)} - \cancelto{\frac{1}{5}}{r}\cdot\cancelto{0}{\tilde u_4^{(1)}} = f(3/2) = 1/\sqrt{2}$$

resulting in the linear system

$$\underbrace{\begin{bmatrix} 7/5 & -1/5 & 0 \\ -1/5 & 7/5 & -1/5 \\ 0 & -1/5 & 7/5 \end{bmatrix}}_{E} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix}^{(1)} = \begin{bmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix}^{(1)} \approxeq \begin{bmatrix} 0.633 \\ 0.895 \\ 0.633 \end{bmatrix}$$

$$\overbrace{\begin{bmatrix} 7/5 & -1/5 & 0 \\ -1/5 & 7/5 & -1/5 \\ 0 & -1/5 & 7/5 \end{bmatrix}} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix}^{(2)} = \begin{bmatrix} 0.633 \\ 0.895 \\ 0.633 \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix}^{(2)} \approxeq \begin{bmatrix} 0.567 \\ 0.801 \\ 0.567 \end{bmatrix}$$

Crank-Nicolson Scheme

Averaging the explicit and implicit method of Richardson to improve error term:

$$g'(x) = \frac{1}{2} \cdot \left(\frac{g(x+\Delta x) - g(x)}{\Delta x} + \frac{g(x) - g(x-\Delta x)}{\Delta x} \right) + \mathcal{O}(\Delta x^2)$$

The Crank-Nicolson values at time-level $k+1$ are computed by solving the system of linear equations

$F^{(n)} \cdot \vec{u}^{(k+1)} = G^{(n)} \cdot \vec{u}^{(k)}$

which can formally be transformed into the linear iteration

$\vec{u}^{(k+1)} = \left(F^{(n)}\right)^{-1} \cdot G^{(n)} \cdot \vec{u}^{(k)}$

where the matrices F and G are for the heat equation $u_t = u_{xx}$:

$F^{(n)} = E^{(n)} + I = \text{tridiag}_{n-1}(-r, 2+2\cdot r, -r)$

$G^{(n)} = C^{(n)} + I = \text{tridiag}_{n-1}(r, 2-2\cdot r, r)$

with

$E^{(n)} = \text{tridiag}_{n-1}(-r, 1+2\cdot r, -r)$ (from implicit)

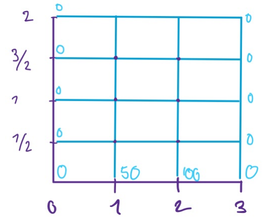
and

$C^{(n)} = \text{tridiag}_{n-1}(r, 1-2\cdot r, r)$ (from explicit)

Example

Given the heat equation $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$ for the domain $\Omega = [0,3] \times [0,\infty[$ and boundary condition $u(x,0) = -25x^2(x-3)$, we want to find $\tilde{u}(x,2)$ using $\Delta x = 1$ and $\Delta t = 0.5$.

Discretisation of Geometry



Solution

With $r = \frac{\Delta t}{\Delta x^2} = \frac{1/2}{1} = 0.5$, we receive the matrices

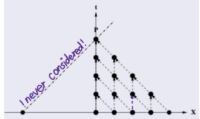
$$F = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 3 \end{bmatrix}, G = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

and thus

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}^{(k+1)} = F^{-1}G \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}^{(k)} \Rightarrow \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}^{(4)} = (F^{-1}G)^4 \begin{bmatrix} \tilde{u}_1 = 50 \\ \tilde{u}_2 = 100 \end{bmatrix}^{(0)}$$

Finite Differences Method for Hyperbolic PDEs

Downwind Scheme



⚠ This scheme will in general diverge and frankly is 🗑.

1. *Discretisation of operators* of PDE $Lu(...) = f(...)$
2. *Discretise geometry* by introducing a grid: $x_{j,k} = (j \cdot \Delta x, k \cdot \Delta t)$ with j as the local index and k as the time index where $k = 0$ is on boundary and $\Delta x = \frac{1}{n}$, $\Delta t = \frac{r}{n}$ and $r = \frac{\Delta t}{\Delta x}$ (⚠ in contrast to $r = \Delta t \div \Delta x^2$ for parabolic equations)
3. *Find matrix $\tilde{u}_{j,k}$* satisfying discretised equation for inner grid points and $\tilde{u}_{j,0} = f_j$ on boundary

Example Using Advection Equation

Given the *advection equation* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$ on $\Omega = (-\infty, \infty) \times [0, \infty)$ with Dirichlet's boundary condition $u(x,0) = f(x) = e^x$. We want to get approximation $\tilde{u}(0,1)$ using $\Delta x = 1/4$ and $\Delta t = 1/4$.

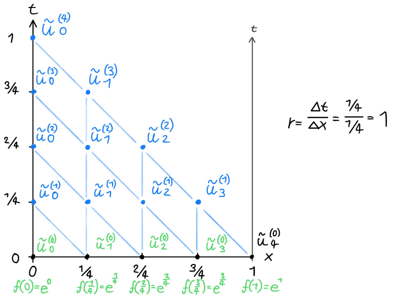
Discretisation of Operators

$$\frac{\tilde{u}(x,t+\Delta t) - \tilde{u}(x,t)}{\Delta t} + \frac{\tilde{u}(x+\Delta x,t) - \tilde{u}(x,t)}{\Delta x} = 0 \quad \Big| \cdot \Delta t$$

$$(\tilde{u}(x_{j,k+1}) - \tilde{u}(x_{j,k})) + r(\tilde{u}(x_{j+1,k}) - \tilde{u}(x_{j,k})) = 0 \quad (\Delta t \div \Delta x = r)$$

$$\Rightarrow \tilde{u}_{j,k+1} = (1+r) \cdot \tilde{u}_{j,k} - r \cdot \tilde{u}_{j+1,k} \quad (= \text{discrete advection eq.})$$

Discretisation of Geometry



Solution Apply discrete advection eq. until $u_0^{(4)}$ is reached.

Upwind Scheme

We use a forward difference in t, just like in the downwind scheme, but a *backward* difference in x.

Example Using Advection Equation

Given the advection equation $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$ on $\Omega = (-\infty, \infty) \times [0, \infty)$ with Dirichlet's boundary condition $u(x,0) = f(x)$ being a box function on the interval $[-1/2, 1/2]$. We want to get approximation $\tilde{u}(x, 3/5)$ using $\Delta x = 2/5$ and $\Delta t = 1/5$.

Discretisation of Operators

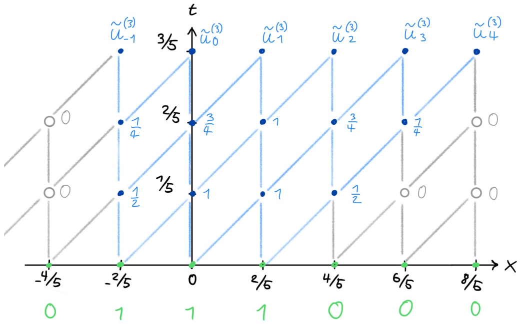
The discrete advection equation for the upwind scheme is derived from

$$\frac{\tilde{u}(x,t+\Delta t) - \tilde{u}(x,t)}{\Delta t} + \frac{\tilde{u}(x,t) - \tilde{u}(x-\Delta x,t)}{\Delta x} = 0$$

and has the iterative form (with $\tilde{u}_j^{(0)} = f_j$ on $\partial\Omega$):

$$\tilde{u}_j^{(k+1)} = (1-r) \cdot \tilde{u}_j^{(k)} + r \cdot \tilde{u}_{j-1}^{(k)}$$

Discretisation of Geometry



Solution Using $\tilde{u}_j^{(k+1)} = 1/2 \cdot \tilde{u}_j^{(k)} + 1/2 \cdot \tilde{u}_{j-1}^{(k)}$

Centred Scheme

Uses a forward difference in t and a centred difference in x.

Example Using Advection Equation

Discretisation of Operators

The discrete advection equation for the centred scheme is derived from the discretised equation:

$$\frac{\tilde{u}(x,t+\Delta t) - \tilde{u}(x,t)}{\Delta t} + \frac{\tilde{u}(x+\Delta x,t) - \tilde{u}(x-\Delta x,t)}{2 \cdot \Delta x} = 0$$

and has the iterative form (with $\tilde{u}_j^{(0)} = f_j$ on $\partial\Omega$):

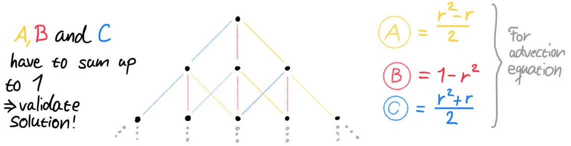
$$\tilde{u}_j^{(k+1)} = -\frac{r}{2} \cdot \tilde{u}_{j+1}^{(k)} + \tilde{u}_j^{(k)} + \frac{r}{2} \cdot \tilde{u}_{j-1}^{(k)}$$

Solution method is analogous to the upwind and downwind scheme, just considering the left and right predecessor as well.

Lax-Wendroff Scheme

Ideally, a second order numerical scheme has the form $\tilde{u}_{j,k+1} = A \cdot \tilde{u}_{i+1,k} + B \cdot \tilde{u}_{i,k} + C \cdot \tilde{u}_{j-1,k}$. Concretely, for the advection equation, the coefficients would be

$$A = \frac{r^2 - r}{2}, \quad B = 1 - r^2, \quad C = \frac{r^2 + r}{2} \quad (r = \frac{\Delta t}{\Delta x})$$



Leapfrog for Wave Equation

Given the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x}$ on $\Omega = [0,1] \times [0, \infty)$ with extra conditions

$$\begin{aligned} u(x,0) &= f(x) \quad x \in [0,1] \\ \tfrac{\partial u}{\partial t}(x,0) &= g(x) \quad x \in [0,1] \\ u(0,t) = u(1,t) &= 1 \quad t \in [0, \infty) \end{aligned}$$

We discretise the geometry again:

$$x_j^{(k)} = (j \cdot \Delta x, k \cdot \Delta t) \text{ with } \Delta x = \frac{1}{n} \text{ and } \Delta t = \frac{r}{n}$$

$$\Rightarrow r = \Delta t \div \Delta x$$

Calculate the first time level by

$$\tilde{u}_j^{(1)} = f(j \cdot \Delta x) + g(j \cdot \Delta x) \cdot \Delta t + f''(j \cdot \Delta x) \cdot \frac{\Delta t^2}{2}$$

afterwards, apply iteratively:

$$\tilde{u}_j^{(k+1)} = r^2 \cdot + \underbrace{\tilde{u}_{j-1}^{(k)}}_{\text{left}} 2 \cdot (1 - r^2) \cdot + \underbrace{\tilde{u}_j^{(k)}}_{\text{center}} r^2 \cdot \underbrace{\tilde{u}_{j+1}^{(k)}}_{\text{right}} - \underbrace{\tilde{u}_j^{(k-1)}}_{\text{center past}}$$

Finite Volume Method for Elliptic PDEs

FVM for Poisson Problems (Voronoi)

> Divergence Theorem: Given a vector field $\vec{f}(x,y)$. The flux integral over the boundary $\partial\Gamma$ satisfies:
>
> $$\oint_{\partial\Gamma} \vec{f}(x,y)\, d\vec{n} = \int_\Gamma \operatorname{div}(\vec{f}(x,y))\, dA \quad (\vec{n} = \text{outer normal vector})$$

Integrating Poisson's equation $\Delta u = f$ over a non-pathological domain $\Gamma \subset \Omega$ leads to

$$\int_\Gamma \Delta u(x,y)\, dxdy = \int_\Gamma f(x,y)\, dxdy \quad \text{(f=0 for Laplacian eq.)}$$

by the divergence theorem, this implies

$$\underbrace{\oint_{\partial\Gamma} \operatorname{grad}(u(x,y))\, d\vec{n}}_{\text{Approx. by Voronoi cells}} = \int_\Gamma f(x,y)\, dxdy$$

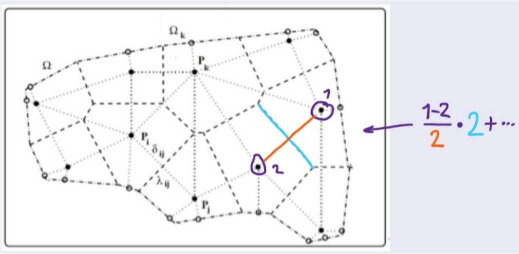
for all non-pathological domains $\Gamma \subset \Omega$.
The *left flux integral* will then be discretised along a finite set of normal derivatives for a discrete, polygonal cell:

$$\oint_{\partial\Gamma} \cdots\, d\vec{n} \approx \sum_j = \frac{u(P_j) - u(P_i)}{\delta_{i,j}} \cdot \lambda_{i,j}$$

for each normal j of cell i with two cell points P_i, P_j and let $\delta_{i,j}$ be the distance between the two cell points and $\lambda_{i,j}$ the length of the common edge of the two cells.
The *right surface integral* needs to be approximated numerically. Easiest way: take function value at Voronoi point and multiply by cell area (obviously not a very good approximation). For the Lapace equation, it's just zero.
We therefore receive a system of linear equations $Av = f$ for the cell values.

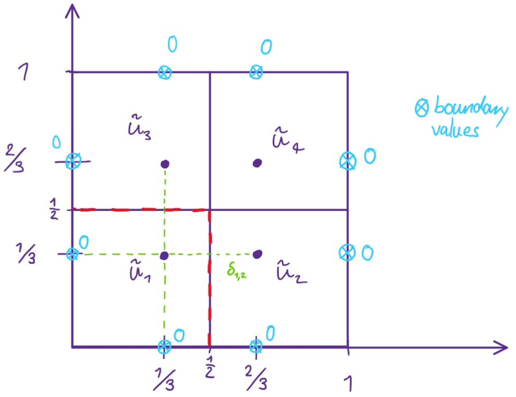
Discretisation of Geometry: Voronoi Cells We can apply the following "algorithm":

1. Replace curved boundary by some polygonal approximation
2. Choose n Voronoi points P_k in Ω and construct for each point its Voronoi cell
3. Compute points that are nearest to the corresponding edge of the boundary where a cell has no close neighbour



Example

Given: The function $u(x,y)$ in $\Omega = [0,1] \times [0,1]$ satisfies $\Delta u(x,y) = 1$. We want to find approximate values in the four points $(1/3, 1/3)$, $(2/3, 1/3)$, $(1/3, 2/3)$ and $(2/3, 2/3)$
Discretisation of Geometry:



Solution
The area of the cell is approximatively $\int_{\Gamma_p} f(x,y)\, dxdy \approx f(P) \cdot h^2$ with cell length h which is always $1/4$ in our case. Therefore, we formulate the linear system:
For $\tilde{u}_1$:
Caution at boundary (⚠): The dashed line usually has different length there. In the example, it stayed the same due to the equidistant points, but for e.g. 1/4 and 3/4 points, it would be 1/4 on boundary and 1/2 for inter-point distances.

$$\frac{\tilde{u}_2 - \tilde{u}_1}{1/3} \cdot \frac{1}{2} + \frac{\tilde{u}_3 - \tilde{u}_1}{1/3} \frac{1}{2} + \underbrace{\frac{0 - \tilde{u}_1}{1/3}}_{⚠} \frac{1}{2} + \underbrace{\frac{0 - \tilde{u}_1}{1/3}}_{⚠} \frac{1}{2} = \iint_{V_1} 1\, dA = \frac{1}{4}$$

$$\frac{3}{2}(\tilde{u}_2 - \tilde{u}_1 + \tilde{u}_3 - \tilde{u}_1 - \tilde{u}_1 - \tilde{u}_1) = \frac{1}{4}$$

$$\tilde{u}_2 + \tilde{u}_3 - 4\tilde{u}_1 = \frac{2}{12} = \frac{1}{6}$$

Due to symmetry, we receive:

$$\left|\begin{array}{ccccc} -4\tilde{u}_1 & +\tilde{u}_2 & +\tilde{u}_3 & & = 1/6 \\ \tilde{u}_1 & -4\tilde{u}_2 & & +\tilde{u}_4 & = 1/6 \\ \tilde{u}_1 & & -4\tilde{u}_3 & +\tilde{u}_4 & = 1/6 \\ & \tilde{u}_2 & +\tilde{u}_3 & -4\tilde{u}_4 & = 1/6 \end{array}\right|$$

with $\tilde{u} = (-1/12, -1/12, -1/12, -1/12)$

Finite Elements

Variational Problem: Method of Ritz

Having a two-point boundary problem $-u''(x) = f(x)$ with $v(0) = v(1) = 0$ on $[0,1]$, we need to find a function that minimises the functional $\phi(v) = \int_0^1 1/2 \cdot v'(x)^2 - f(x) \cdot v(x)\, dx$

Idea of Ritz: Focusing on a vector subspace $\mathbb{V}^{(n)} \subset \mathbb{V}$ in which we know that there's a solution and then minimise the functional for all linear combinations:

$$a_1 v_1(x) + \ldots + a_n v_n(x), \quad a_k \in \mathbb{R}$$

We have the Ritz matrix

$$\begin{aligned} R_{j,k}^{(n)} &= \int_0^1 v_j'(x) \cdot v_k'(x)\, dx \\ &= v_j'(x) \cdot v_k(x)|_0^1 - \int_0^1 v_j''(x) \cdot v_k(x)\, dx \\ &= -\int_0^1 v_j''(x) \cdot v_k(x)\, dx \quad \text{(homogeneous b.c.)} \end{aligned}$$

and the Ritz Vector

$$r_k^{(n)} = \int_0^1 f(x) \cdot v_k(x)\, dx$$

to find the a coefficients: $R^{(n)} \cdot \vec{a} = \vec{r}^{(n)}$.

Example

Given Baby Poisson $-u''(x) = f(x)$ with $f(x) = \pi^2 \sin(\pi x)$ with $u(0) = u(1) = 0$. We want to compute the coefficient b in the ansatz $\tilde{u}(x) = b \cdot (x - 2x^2 + x^3)$.

Solution We compute the (1-dimensional) Ritz matrix R with $v1(x) = x - 2x^2 + x^3$:

$$R_{11} = \int_0^1 {v_1}'(x) \cdot {v_1}'(x)\, dx = \int_0^1 (1 - 4x + 3x^2)^2\, dx = 2/15$$

and the (1-dimensional) Ritz vector with

$$r_1 = \int_0^1 f(x) \cdot v_1(x)\, dx = \int_0^1 \pi^2 \sin(\pi x) \cdot (x - 2x^2 + x^3)\, dx = \frac{2}{\pi}$$

Thus, we get $R_{11} \cdot b = r_1$ and therefore $b = 15/\pi$.

Method of Galerkin

Weak reformulation: Find a function $u(x)$ (for problem $-u''(x) = f(x) \Rightarrow u''(x) + f(x) = 0$) in $\mathbb{V}$ such that for all $v(x) \in \mathbb{V}$ one has

$$\int_0^1 \big(u''(x) + f(x)\big) \cdot v(x)\ dx = 0$$

and then focus on n carefully chosen functions $v_1(x), \dots, v_n(x)$ and find a function $\tilde{u}(x)$ in $\mathbb{V}^{(n)}$, called ansatz, (that already satisfy the Dirichlet boundary conditions) such that, when substituted for the above integral

$$\int_0^1 \big(\tilde{u}''(x) + f(x)\big)\, v_k(x)\ dx = 0, \quad k = 1, \dots, n$$

The n equations thus turn into n linear equations

$$\int_0^1 \big(a_1 \cdot v_1''(x) + \ldots + a_n \cdot v_n''(x) + f(x)\big) \cdot v_k(x)\ dx = 0$$

for $k = 1, \dots, n$, giving the Galerkin Matrix $G^{(n)}_{k,j} = \int_0^1 v_j''(x) \cdot v_k(x)\ dx$ and the Galerkin vector $g^{(n)}_k = \int_0^1 f(x) \cdot v_k(x)\ dx$ and the system $G^{(n)} \cdot \vec{a} + \vec{g}^{(n)} = 0$. We can also observe that $G^{(n)} = -R^{(n)}$ and $\vec{g}^{(n)} = \vec{r}^{(n)}$.

Example

Given The function $u(x)$ on $\Omega = [0, 1]$ satisfies Helmholtz's equation $u''(x) + 17u(x) = 0$ (*$=$ 0 important, move to left side*) with Dirichlet conditions $u(0) = 0, u(1) = 1$. Determine an approx. function $\tilde{u}(x)$ for $u(x)$ with the ansatz $\tilde{u}(x) = x + a_1(x - x^2) + a_2(x^2 - x^3)$ (that already satisfies b.c.)
Solution Given the ansatz

$$\tilde{u}(x) = x + a_1(x - x^2) + a_2(x^2 - x^3)$$

$$\tilde{u}''(x) = -2a_1 + a_2(2 - 6x)$$

$$v_1(x) = x - x^2$$

$$v_2(x) = x^2 - x^3$$

we receive the linear system

$$\int_0^1 \big(\tilde{u}''(x) + 17\tilde{u}(x)\big) \cdot v_1(x)\ dx = 0$$

$$\int_0^1 \big(\tilde{u}''(x) + 17\tilde{u}(x)\big) \cdot v_2(x)\ dx = 0$$

which gives

$$\int_0^1 \left[-2a_1 + a_2(2 - 6x) + 17(x + a_1(x - x^2) + a_2(x^2 - x^3))\right] \cdot (x - x^2)\ dx = 0$$

$$\int_0^1 \left[-2a_1 + a_2(2 - 6x) + 17(x + a_1(x - x^2) + a_2(x^2 - x^3))\right] \cdot (x^2 - x^3)\ dx = 0$$

separating by coefficients (e.g. for first equation):

$$\int_0^1 \left[a_1(-2 + 17x - 17x^2) + a2(2 - 6x + 17x^2 - 17x^3)\right] \cdot (x - x^2)\ dx = 0$$

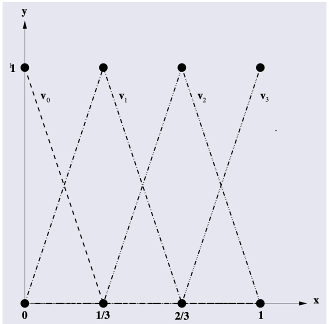
leaves a system in the form $a_1 \int_0^1 \cdots + a_2 \int_0^1 \cdots + \int_0^1 \cdots = 0$.
With the integrals solved, we get:

$$a_1 \frac{7}{30} + a_2 \frac{7}{60} + \frac{17}{12} = 0$$

$$a_1 \frac{7}{60} + a_2 \frac{1}{85} + \frac{17}{12} = 0$$

and we receive $a_1 \approx -8.45$ and $a_2 \approx 4.76$ and thus $\tilde{u}(x) = x - 8.45(x - x^2) + 4.76(x^2 - x^3)$.

Elliptic



Introduce on problem domain nodal points that divide the geometry into cells or meshes. Associate to each nodal variable a_k of a nodal point a local function v_k from a set of given local basis functions $v_1(x), \dots, v_n(x)$ that are continuous and piecewise differentiable. Thus, the ansatz $\tilde{u}(x) = a_0 v_0(x) + a_1 v_1(x) + \ldots + a_n v_n(x)$ has shape functions v_k that are one on the respective nodal point k. Then, we use shape functions to represent the PDE on the mesh, e.g. using triangular functions $l_1(x) = 1 - x$ and $l_2(x) = x$ to obtain *local* element matrices

$$E_{\text{step}} = \begin{bmatrix} \int_0^1 l_1'(s) \cdot l_1'(s)\ ds & \int_0^1 l_1'(s) \cdot l_2'(s)\ ds \\ \int_0^1 l_2'(s) \cdot l_1'(s)\ ds & \int_0^1 l_2'(s) \cdot l_2'(s)\ ds \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

that can then be used to construct the mesh matrix $M = 1/h \cdot E$ using mesh size h. Afterwards, the global Ritz matrix can be computed, e.g. for a one-dimensional mesh with 4 nodal points and 2 unknown inner points, yielding 4 base functions v_1, v_2, v_3, v_4:

$$R^4 = \begin{bmatrix} M^{(1)}_{0,0} & M^{(1)}_{0,1} & 0 & 0 \\ M^{(1)}_{1,0} & M^{(1)}_{1,1} + M^{(2)}_{0,0} & M^{(2)}_{0,1} & 0 \\ 0 & M^{(2)}_{1,0} & M^{(2)}_{1,1} + M^{(3)}_{0,0} & M^{(3)}_{0,1} \\ 0 & 0 & M^{(3)}_{1,0} & M^{(3)}_{1,1} \end{bmatrix}$$

The system vector can then be calculated:

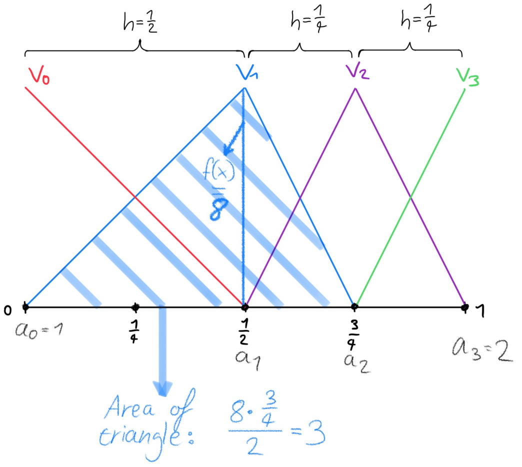
$$\vec{r^4} = \begin{pmatrix} \int_0^1 f(x) \cdot v_0(x)\ dx \\ \vdots \\ \int_0^1 f(x) \cdot v_3(x)\ dx \end{pmatrix}$$

Finally, we have obtained the Ritz system: $R^4 \cdot \vec{a} = \vec{r^4}$. That yields the approximation function (as defined by the ansatz): $\tilde{u}(x) = \sum_{i=0}^{3} a_i \cdot v_i(x)$ with a_0 and a_3 fulfilling the boundary conditions.
This shape function approach can also be applied to the Ritz vector. We get $\tilde{f}(x) = f(a_0) \cdot v_0(x) + \ldots + f(a_n) v_n(x)$ and we approximate $r^n_k = \int_0^1 f(x) \cdot v_k(x)\ dx$ by $\tilde{r}^n_k = \int_0^1 \tilde{f}(x) \cdot v_k(x)\ dx$ which essentially is $\tilde{r}^n = S^n \cdot \vec{f^n}$ where $\vec{f^n}$ is the vector of nodal values and $S^n_{k,j} = \int_0^1 v_j(x) \cdot v_k(x)\ dx$

Example With Inhomogeneous B.C.

Given A real function $u(x)$ on interval $\Omega = [0, 1]$ that satisfies the differential equation $u''(x) + 8 = 0$ and the boundary conditions $u(0) = 1$ and $u(1) = 2$. We want to find an approximation function $\tilde{u}(x)$ using the meshes $[0, 0.5], [0.5, 0.75], [0.75, 1]$ and linear shape functions.

Discretisation of Geometry



Solution

We get the following mesh matrices:
Mesh 1 (step size = 1/2):

$$\frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = {}^1\!/\!_2{}^{-1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

Mesh 2 (step size = 1/4):

$$\frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

Mesh 3 (step size = 1/4):

$$4\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

resulting in the global Ritz-Matrix and Ritz-Vector:

$$\overbrace{\begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 + 4 = 6 & -4 & 0 \\ 0 & -4 & 4 + 4 = 8 & -4 \\ 0 & 0 & -4 & 4 \end{bmatrix}}^{R} \overbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}}^{\vec{a}} = \overbrace{\begin{bmatrix} \int_0^1 f(x) v_0(x)\ dx \\ \int_0^1 f(x) v_1(x)\ dx \\ \int_0^1 f(x) v_2(x)\ dx \\ \int_0^1 f(x) v_3(x)\ dx \end{bmatrix}}^{\vec{r}}$$

$$= \begin{bmatrix} \frac{1/2 \cdot 8}{2} = 2 & \text{(area of red shape function)} \\ \frac{3/4 \cdot 8}{2} = 3 & \text{(area of blue shape function)} \\ \frac{1/2 \cdot 8}{2} = 2 & \text{(area of violet shape function)} \\ \frac{1/4 \cdot 8}{2} = 1 & \text{(area of green shape function)} \end{bmatrix}$$

to satisfy inhomogeneous boundary conditions, we set a_0 and a_3 to the boundary condition in the vector $\vec{a}$ and eliminate the corresponding equations, leading to the reduced Ritz system:

$$\begin{bmatrix} -2 & 6 & -4 & 0 \\ 0 & -4 & 8 & -4 \end{bmatrix} \begin{bmatrix} 1 \text{ (b.c.)} \\ a_1 \\ a_2 \\ 2 \text{ (b.c.)} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

meaning

$$\begin{bmatrix} 6 & -4 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} -2 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

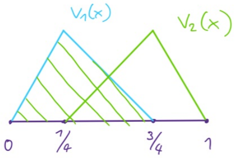
giving $a_1 = {}^5\!/_2$ and $a_2 = {}^5\!/_2$.

Example With Homogeneous Boundary Conditions

Given the differential equation $u''(x) + 1 = 0$ on $\Omega = [0, 1]$ with homogeneous boundary conditions. We want to approximate using meshes $[0, 0.25], [0.25, 0.75], [0.75, 1]$.

Discretisation of Geometry

Since we have homogeneous boundary conditions, we can ignore v_0 and v_3:



giving $\tilde{u}(x) = 0v_0(x) + a_1v_1(x) + a_2v_2(x) + 0v_3(x)$.

Solution

For steps 1, 2 and 3, we get the mesh matrices

$$M_1 = 4 \cdot E, M_2 = 2 \cdot E, M_3 = 4 \cdot E$$

using element matrix E of step function and thus the Ritz matrix

$$R = \begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 6 & -2 & 0 \\ 0 & -2 & 6 & -1 \\ 0 & 0 & -4 & 4 \end{bmatrix}$$

due to homogeneous boundary conditions, we cancel the first and last row and columns, resulting in the reduced Ritz system:

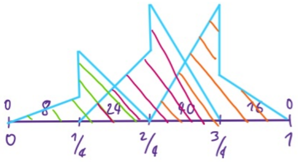
$$\begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} {}^3\!/_8 \\ {}^3\!/_8 \end{bmatrix}$$

Example With Funky f(x)

Problem: $-u''(x) = f(x)$ in $\Omega = [0, 1]$ with

$$f(x) = \begin{cases} 8 & \text{if } x \in [0, {}^1\!/_4), \\ 24 & \text{if } x \in [{}^1\!/_4, {}^1\!/_2), \\ 40 & \text{if } x \in [{}^1\!/_2, {}^3\!/_4), \\ 16 & \text{if } x \in [{}^3\!/_4, 1), \end{cases}$$

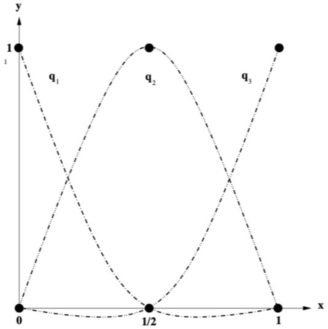
and homogeneous b.c. and $h = 1/4$.



The Ritz vector then is:
For v_1: $8 \cdot \frac{1}{4}\frac{1}{2} = 1$; $24 \cdot \frac{1}{4}\frac{1}{2} = 3 \to 1 + 3 = 4$
For v_2: $24 \cdot \frac{1}{4}\frac{1}{2} = 3$; $40 \cdot \frac{1}{4}\frac{1}{2} = 5 \to 3 + 5 = 8$
For v_3: $40 \cdot \frac{1}{4}\frac{1}{2} = 5$; $16 \cdot \frac{1}{4}\frac{1}{2} = 2 \to 5 + 2 = 7$

p-Strategy

Using quadratic shape functions
$q_1(x) = (1 - x)(1 - 2x), q_2(x) = 4x(1 - x), q_3(x) = -x(1 - 2x)$ as shape functions



The v-functions for three nodes then look like

$$\begin{array}{rcrrr} v_0(x) & = & q_1(3x); & 0; & 0, \\ v_1(x) & = & q_2(3x); & 0; & 0, \\ v_2(x) & = & q_3(3x); & q_1(3x-1); & 0, \\ v_3(x) & = & 0 & q_2(3x-1); & 0, \\ v_4(x) & = & 0; & q_3(3x-1); & q_1(3x-2), \\ v_5(x) & = & 0; & 0; & q_2(3x-2), \\ v_6(x) & = & 0; & 0; & q_3(3x-2). \end{array}$$

with an affine transformation of the x values onto the nodal points.
We get a new element matrix:

$$E = \frac{1}{3} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

and the associated mesh matrix $M = 1/h \cdot E$. The Ritz vector can be computed analogous to the previous methods:

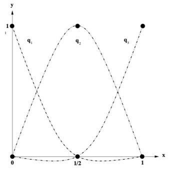
$$\vec{r} = \begin{bmatrix} \int_0^1 f(x) \cdot v_0(x)\ dx \\ \int_0^1 f(x) \cdot v_1(x)\ dx \\ \vdots \\ \int_0^1 f(x) \cdot v_n(x)\ dx \end{bmatrix}$$

with the difference that the integral is not as straight forward as with the triangle function (ℹ except for baby Laplace problem $u''(x) = 0$, since there the Ritz-vector is zero and no integration needed)

Example

Given $-u''(x) = -2$ on $\Omega = [0, 1]$ with $u(0) = 1$ and $u(1) = 3$. We want to approximate with $\tilde{u}$ and $h = 1$.

Discretisation of Geometry The resulting curves are exactly



where q_1 and q_3 are determined by the boundary conditions.

Solution We get the mesh matrix and a resulting system

$$\frac{1}{h = 1} \frac{1}{3} \begin{bmatrix} \{7 & -8 & 1\} \\ -8 & 16 & -8 \\ \{1 & -8 & 7\} \end{bmatrix} \begin{bmatrix} 1 \\ a \\ 3 \end{bmatrix} = \begin{bmatrix} \times \\ \int_0^1 f(x) \cdot q_2(x)\ dx \\ \times \end{bmatrix}$$

We only need to evaluate the integral of the q_2 since the other curves are determined already. We receive:

$$\int_0^1 -2 \cdot (4x - 4x^2)\ dx = -\frac{4}{3}$$

and since we can eliminate the first and second row, we get:

$$\frac{1}{3}(-8 + 16a - 24) = -\frac{4}{3} \quad \Rightarrow \quad a = \frac{7}{4}$$
$$\Rightarrow \tilde{u}(x) = 1 \cdot q_1(x) + {}^7\!/_4 \cdot q_2(x) + 3q_3(x)$$

Formulas and Basic Math

Roots

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a}$$

Logarithm

$$\log_n(a \cdot b) = \log_n(a) + \log_n(b)$$

$$\log_n(a \div b) = \log_n(a) - \log_n(b)$$

$$\log_n(a^b) = b \cdot \log_n(a)$$

Quadratic Fromula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

With discriminant $b^2 - 4ac$:

1. $b^2 - 4ac > 0$, there are *two distinct real solutions*
2. $b^2 - 4ac = 0$, *there is* one real solution
3. $b^2 - 4ac < 0$, there are *no real solutions*

Trigonometry

“Normal”

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$1 \div \cot(x) = \tan(x)$$

$$\sin -\theta = -\sin\theta \text{ (cos and tan same)}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = 2\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Hyperbolic

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x}$$

$$\tanh x = \sinh x \div \cosh x$$

$$\coth x = \cosh x \div \sinh x$$

$$\text{sech } x = 1 \div \cosh x = 2 \div (e^x + e^{-x})$$

$$1 = \cosh^2(x) - \sinh^2(x)$$

$$-1 = \sinh^2(x) - \cosh^2(x)$$

$$e^x = \cosh(x) + \sinh(x)$$

$$e^{-x} = \cosh x - \sinh x$$

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$$

$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$

$$\sinh(2x) = 2 \cdot \sinh(x)\cosh(x)$$

$$\sqrt{x^2 + 1} = \cosh(\text{arcsinh}(x))$$

$$\sqrt{x^2 - 1} = \sinh(\text{arccosh}(x))$$

Derivatives

$f(x)$	$\frac{df}{dx}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\text{arcsinh}(x)$	$1 \div \sqrt{x^2 + 1}$
$\text{arccosh}(x)$	$1 \div \sqrt{x^2 - 1}$ $(1 < x)$
$\tan(x)$	$\cos^{-2}(x)$
$\log(x)$	x^{-1}

Integrals

$$\int x^n \, dx = \tfrac{1}{n+1} x^{n+1} + C$$

$$\int \tfrac{1}{x} \, dx = \ln|x| + C$$

$$\int \tfrac{1}{ax+b} \, dx = \tfrac{1}{a} \ln|ax + b| + C$$

$$\int \tfrac{1}{(x+a)^2} \, dx = -\tfrac{1}{x+a} + C$$

$$\int \tfrac{1}{1+x^2} = \tan^{-1} x + C$$

$$\int \ln ax \, dx = x \ln ax - x + C$$

$$\int e^{ax} \, dx = \tfrac{1}{a} e^{ax} + C$$

$$\int \sin(ax) \, dx = -\tfrac{1}{a}\cos(ax) + C$$

$$\int \sin^2(ax) \, dx = \tfrac{x}{2} - \tfrac{\sin(2ax)}{4a} + C$$

$$\int x\cos x \, dx = \cos x + x\sin x + C$$

$$\int \sinh(ax) \, dx = a^{-1}\cosh ax + C$$

$$\int \cosh(ax) \, dx = a^{-1}\sinh ax + C$$

Integration Techniques

Integration by Parts

$$\int_a^b u(x)v'(x) \, dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x) \, dx$$

Or, with $u = u(x)$, $du = u'(x) \, dx$, $v = v(x)$ and $dv = v'(x) \, dx$:

$$\int u \, dv = uv - \int v \, du$$

Substitution

$$\int_a^b f(g(x)) \cdot g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

Leibniz Integral Rule

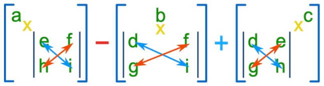
$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t) \, dt\right) =$$

$$f(x, b(x)) \cdot \frac{d}{dx} b(x) - f(x, a(x)) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t) \, dt$$

Special case where $a(x) = a = \text{const.}$ and $b(x) = b = \text{const.}$:

$$\frac{d}{dx}\left(\int_a^b f(x,t) \, dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t) \, dt$$

Determinant



$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Properties

- $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$
- $\det(A^T) = \det(A)$
- $\det(I) = 1$
- $\det(cA) = c^n \det(A)$ (for an $n \times n$ matrix)
- $\det(AB) = \det(A)\det(B)$
- $\det(A) = \prod_{i=1}^n \lambda_i$

Particular Solutions to Simple ODEs

$$f'(x) = \tfrac{c}{x} f(x) \Rightarrow f(x) = k_1 y^c$$

$$f'(x) = c \cdot f(x) \Rightarrow f(x) = k_1 e^{cx}$$

$$f''(x) = c \cdot f(x) \Rightarrow f(x) = k_1 e^{\sqrt{c}x} + k_2 e^{-\sqrt{c}x}$$

$$f''(x) = -c \cdot f(x) \Rightarrow f(x) = k_1 \sin(\sqrt{c}x) + k_2 \cos(\sqrt{c}x)$$

$$f'(x) + af(x) = b \Rightarrow f(x) = \left(f(0) - \tfrac{b}{a}\right) e^{-ax} + \tfrac{b}{a}$$

Harmonic Function

A function is harmonic if it fulfils $\Delta f = 0$. The mean value property applies:

$$u(x) = \frac{1}{\mu(S_r(x))} \int_{S_r(x)} u(y) \, d\mu(y)$$

Polar Coordinates

$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$r = \sqrt{x^2 + y^2} \quad \text{(!) when converting } x^2 + y^2 \text{, it's } r^2\text{!}$$

$$\varphi = \text{atan2}\left(\frac{y}{x}\right)$$

The Laplace operator in polar coordinates is

$$\begin{aligned} \Delta u &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} \\ &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} \end{aligned}$$

Tridiagonal Matrix

$$M = \text{tridiag}_n(a, b, c) = \begin{bmatrix} b & c & & & & \\ a & b & c & & & \\ & a & b & c & & \\ & & \ddots & \ddots & \ddots & \\ & & & a & b & c \\ & & & & a & b \end{bmatrix}_{n \times n}$$